Japanese Grant Aid for Human Resource Development Scholarship (JDS)
Advanced Mathematics Aptitude Test
2024

Prepared by Japanese Development Service Co., Ltd.

Instructions:

This test has 3 questions. Answer all questions.
The total mark for this test is 42.

Reduce square roots（$\sqrt{\ }$） as much as possible. (Example: Express $\sqrt{48}$ as $4\sqrt{3}$ not as $2\sqrt{12}$ *or* $\sqrt{48}$.)

For fractions, reduce the fraction to its lowest terms. (Example: Substitute $\frac{1}{3}$ for $\frac{2}{6}$.)

Please show all your work, including intermediate steps. You may also write your answers in the “memo” section.

10. Let us roll one fair die three times. Let the number that comes up on the first roll be **a**, the number on the second roll be **b**, and the number on the third roll be **c**.

(a) Find the probability that the product **abc** is a multiple of 3. (3 marks)

"The product **abc** is a multiple of 3" means "at least one of **a, b**, and **c** is a multiple of 3."

If we consider the complementary event, it becomes easy.

The condition for the product **abc** to not be a multiple of 3 is that **a, b**, and **c** are all not multiples of 3.

The probability that **a** is not a multiple of 3 is $1-\frac{2}{6}=\frac{2}{3}$ (1 mark for stating the probability that **a** is not a multiple of 3)

The same goes for the probability that **b** and **c** are not multiples of 3.

Therefore, the probability we are looking for is

$$1-\left(\frac{2}{3}\right)^3=1-\frac{8}{27}=\frac{19}{27}$$

(1 mark for working out, 1 mark for correct final answer)

(3 marks total)

(b) Find the probability that the value of the limit $\lim_{x\to-1}\frac{ax^2+bx+c}{x+1}$ exists. (7 marks)

Firstly, the denominator $\lim_{x\to-1}(x+1)=0$

Check if the numerator also becomes zero. The limit will exist if the numerator becomes zero when x = -1: (1 mark for statement of conditions when the limit exists)

$ax^2+bx+c=a\times(-1)^2+b\times(-1)+c$, so $a-b+c=0$ — ① (1 mark for equation)

① becomes $c = -a + b$, then $\frac{ax^2+bx+c}{x+1} = \frac{ax^2+bx+(-a+b)}{x+1}$

$$= \frac{a(x^2-1)+b(x+1)}{x+1}$$

$$= \frac{a(x+1)(x-1)+b(x+1)}{x+1}$$

$$= \frac{(x+1)\{a(x-1)+b\}}{x+1}$$

$$= a(x-1) + b$$

Therefore $\lim_{x \to -1} \frac{ax^2+bx+c}{x+1} = \lim_{x \to -1} \{a(x-1) + b\} = -2a + b$, so the limit exists.

(2 marks for working to show when the limit exists)

$a - b + c = 0$ from ① becomes $b = a + c$ —②

Now, if we look at $b = a + c$ in ②, **a, b,** and **c** are the results of the dice rolls.

Therefore, **b** cannot be 1. $b \neq 1$.

$b \neq 0$ as well.

Divide $b = a + c$ into cases:

i) if $b = 2$, $(a, c) = (1,1)$ → 1 way

ii) if $b = 3$, $(a, c) = (1,2), (2,1)$ → 2 ways

iii) if $b = 4$, $(a, c) = (1,3), (2,2), (3,1)$ → 3 ways

iv) if $b = 5$, $(a, c) = (1,4), (2,3), (3,2), (4,1)$ → 4 ways

v) if $b = 6$, $(a, c) = (1,5), (2,4), (3,3), (4,2), (5,1)$ → 5 ways

From the above, the values of **a, b,** and **c** that satisfy ① $a - b + c = 0$ are $1 + 2 + 3 + 4 + 5 = 15$ combinations. (1 mark for working to calculate number of combinations, 1 mark for correct number of combinations)

The total number of possible outcomes when rolling the dice three times is $6 \times 6 \times 6 = 216$

Therefore, the probability we are looking for is $\frac{15}{216} = \frac{5}{72}$ (1 mark for correct final answer)

(c) In a cubic function $f(x) = x^3 + ax^2 + bx + c$, find the probability that the function $y = f(x)$ takes both the maximum value and the minimum value. (6 marks)

Differentiating $f(x)$, $f'(x) = 3x^2 + 2ax + b$ (1 mark for differentiation)

The function y = $f(x)$ takes maximum values and minimum values when $f'(x) = 0$ takes two distinct real solutions. (1 mark for using the condition taking both maximum and minimum values)

Therefore, the discriminant of $f'(x)$ must be positive.

$\Delta = (2a)^2 - 4(3)(b) = 4a^2 - 12b > 0$ therefore $b < \frac{a^2}{3}$ (1 mark for the inequality)

Therefore, the pair (a, b) that meets the above is

If $a = 2,\ b = 1$
If $a = 3,\ b = 1, 2$
If $a = 4,\ b = 1, 2, 3, 4, 5$
If $a = 5,6,\ b = 1, 2, 3, 4, 5$ 6, in total 20 pairs. (1 mark for working out the pairs, 1 mark for correct number of pairs)

six ways of **c** (c = 1, 2, 3, 4, 5, and 6) correspond to these 20 pairs. Therefore, the probability we are looking for is (1 mark for correct final answer)

$$\frac{20 \times 6}{6^3} = \frac{5}{9}$$

11. (a) For two complex numbers $\alpha = 1 + i$, and z, how does point αz move relative to point

z? (4 marks)

When we represent the complex number $\alpha = 1 + i$ in polar form, we get:

$$\alpha = 1 + i = \sqrt{\boxed{A}}\left(\cos\frac{\boxed{B}}{\boxed{C}}\pi + i\sin\frac{\boxed{B}}{\boxed{C}}\pi\right).$$

We observe that point α z is the point obtained by rotating point z by $\frac{\boxed{D}}{\boxed{E}}\pi$ around the origin and then scaling its distance by a factor of $\sqrt{\boxed{E}}$.

Fill in the blanks below:

$$\sqrt{\boxed{A}} = \sqrt{\square},\ \frac{\boxed{B}}{\boxed{C}} = \frac{\square}{\square},\ \frac{\boxed{D}}{\boxed{E}}\pi = \frac{\square}{\square}\pi,\ \sqrt{\boxed{F}} = \sqrt{\square},$$

A = 2, B = 1, C = 4, D = 1, E = 4, F = 2,

The distance r from the origin to z is $r = |1 + i| = \sqrt{1^2 + 1^2} = \sqrt{2}$ as the absolute value of the complex number. (1 mark for modulus)

$$\alpha = 1 + i = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right) \quad \text{due to } \frac{1}{\sqrt{2}} = \cos\frac{\pi}{4} = \sin\frac{\pi}{4},$$

$$= \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right).$$ (1 mark for correct polar form)

This is the polar form of the complex number $\alpha = 1 + i$,

and the angle from the real axis to the vector is $\theta = \frac{\pi}{4}$

(1 mark for values of A, B, C; 1 mark for values of D, E, F)

(b) Represent the following complex numbers in polar form. (In the following, the range of an argument θ is $0 \leqq \theta < 2\pi$.) (4 marks)

(i) $1 + \sqrt{3}i$ (ii) -1

(b)(i) Modulus of the complex number $1 + \sqrt{3}i$ is

$$|1 + \sqrt{3}i| = \sqrt{1^2 + (\sqrt{3})^2} = 2$$

The argument of the complex number θ is

$$cos\theta = \frac{1}{2}, sin\theta = \frac{\sqrt{3}}{2} \text{ then } \theta = \frac{\pi}{3}$$

therefore $1 + \sqrt{3}i = 2\left(cos\frac{\pi}{3} + i\ sin\frac{\pi}{3}\right)$ (1 mark for working, 1 mark for correct answer)

(b)(ii) Modulus of the complex number -1 is

$$|-1| = \sqrt{(-1)^2 + 0^2} = 1$$

The argument of the complex number θ is

$$cos\theta\frac{-1}{1} = -1,\ sin\theta\frac{0}{1} = 0, \text{ then } \theta = \pi$$

therefore $-1 = cos\pi + i\ sin\pi$

(1 mark for working, 1 mark for correct answer)

(c) Convert the complex numbers $z = \frac{\sqrt{3}-i}{1-i}$ into the form $a + bi$. Then, find the value of $cos\frac{\pi}{12}$, $sin\frac{\pi}{12}$ by representing the denominator and numerator in polar form. (8 marks)

(c) Rationalize the denominator of $z = \frac{\sqrt{3}-i}{1-i}$

$$= \frac{(\sqrt{3}-i)(1+i)}{(1-i)(1+i)}$$

$$= \frac{(\sqrt{3}+1) + i(\sqrt{3}-1)}{2}$$

$$= \frac{\sqrt{3}+1}{2} + \frac{\sqrt{3}-1}{2}i \quad —①$$

(1 mark for working – rationalizing, 1 mark for correct rationalized form)

Then, convert the denominator and numeral of z into polar form.

The absolute value of numerator is $\sqrt{\left(\sqrt{3}\right)^2 + (-1)^2} = 2$, so if we bring it to the front, we get (numerator) $= 2(\frac{\sqrt{3}}{2} - \frac{1}{2}\,i)$

$$= 2\left(cos\frac{11}{6}\pi + i\,sin\frac{11}{6}\pi\right)$$

(1 mark for numerator in polar form working, 1 mark for correct polar form)

The absolute value of denominator is $\sqrt{(1)^2 + (-1)^2} = \sqrt{2}$, so if we bring it to the front,

we get (denominator) $= \sqrt{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)$

$$= \sqrt{2}\left(cos\frac{7}{4}\pi + i\,sin\frac{7}{4}\pi\right)$$

From the above results $\mathbf{Z} = \dfrac{2\left(\cos\frac{11}{6}\pi + \mathrm{i}\sin\frac{11}{6}\pi\right)}{\sqrt{2}\left(\cos\frac{7}{4}\pi + \mathrm{i}\sin\frac{7}{4}\pi\right)}$

$$= \sqrt{2}\left\{\left(cos(\frac{11}{6}\pi - \frac{7}{4}\pi) + i\,sin(\frac{11}{6}\pi - \frac{7}{4}\pi\right)\right\}$$

$$= \sqrt{2}\left(cos\frac{\pi}{12} + i\,sin\frac{\pi}{12}\right) = \sqrt{2}cos\frac{\pi}{12} + i\sqrt{2}sin\frac{\pi}{12} \quad \text{—②}$$

(1 mark for denominator in polar form working, 1 mark for correct polar form)

Now, comparing the real part and imaginary part of ① and ②,

$$\sqrt{2}cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2} \quad \rightarrow \quad cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{4}$$

$$\sqrt{2}sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2} \quad \rightarrow \quad sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{4}$$

(1 mark for correct value of cos, 1 mark for correct value of sin)

12 . (a) Find the range of values of **m** such that the curve $y = x\left(x - 3\right)^2$ and straight line $y = mx$ $(x \geqq 0)$ intersect at three different points on the xy plane. (4 marks)

Factorize $x^3 - 6x^2 + 9x = mx$ which gives $x(x^2 - 6x + (9 - m)) = 0$

Therefore $x = 0$ or $x^2 - 6x + (9 - m) = 0$ −① (1 mark for equating two lines)

If ① has two distinct positive solutions, the curve and straight line intersect at three different points at $x \geqq 0$. Discriminant of ① will be positive.

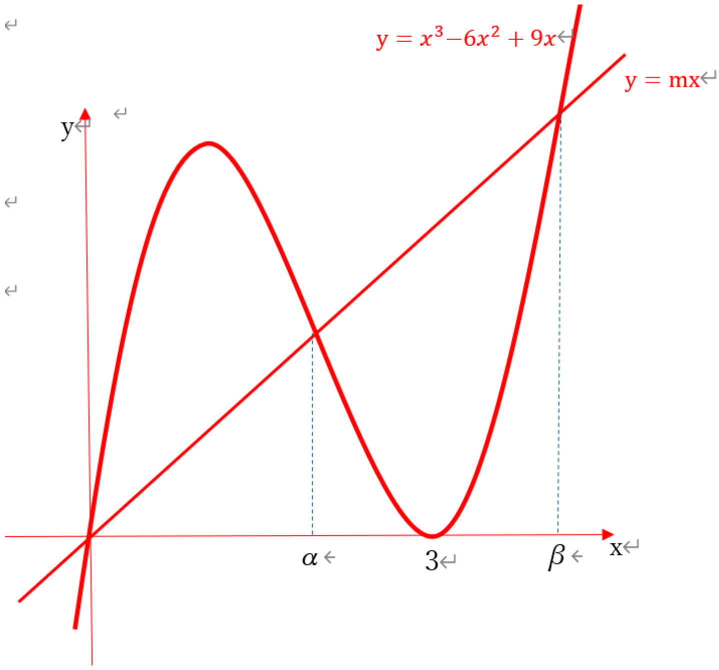
$$\Delta = (-6)^2 - 4(1)(9 - m) = 36 - 36 + 4m = 4m > 0$$

Therefore m > 0. (1 mark for use of discriminant)

The conditions are $m > 0$ and $9 - m > 0$.

Therefore $0 < m < 9$.

(2 marks for correct answer: correct upper and lower bound)



(b) Find the value of **m** such that the two areas of the region bounded by the curve and the straight line are equal. (4 marks)

Let the x-coordinates of the three intersections of the curve and the straight line be 0, α , and β （$0 < \alpha < \beta$）.

When the two areas of shapes surrounded by a curve and a straight line are equal,

$$\int_0^\alpha \left\{x\left(x-3\right)^2 - mx\right\} dx = \int_\alpha^\beta \left\{mx - x\left(x-3\right)^2\right\} dx$$

$$\int_0^\beta \left\{x\left(x-3\right)^2 - mx\right\} dx = 0$$

$$\int_0^\beta \left(x^3 - 6x^2 + 9x - mx\right) dx = 0$$

$$\left[\frac{1}{4}x^4 - 2x^3 + \frac{9}{2}x^2 - \frac{m}{2}x^2\right]_0^\beta = 0$$

$$\frac{1}{4}\beta^4 - 2\beta^3 + \frac{9-m}{2}\beta^2 = 0$$

(1 mark for correctly equating regions of the curve, 1 mark for working)

As $\beta \neq 0$, Divide both sides by β^2 and simplify,

$$\beta^2 - 8\beta + 2(9-m) = 0 \quad -②$$

(1 mark for correct quadratic)

$x = \beta$ is from the solution of $x^2 - 6x + 9 - m = 0$

$$\beta^2 - 6\beta + 9 - m = 0$$

Substituting this into ② and rearranging it, we get $9 - m = 2\beta$ —③

Substituting ③ in ②, $\beta^2 - 4\beta = 0$

As $\beta \neq 0$, $\beta = 4$. (1 mark for correct value of β)

$$\therefore m = 9 - 2\beta$$

Then substituting $\beta = 4$ into this.

$m = 1$ (Satisfies $0 < m < 9$) (1 mark for correct value of m)

(c) Find the sum of the two areas. (2 marks)

(1 mark for working, 1 mark for correct final answer)

Area S which we will find is, by using (b)

$$S=2\int_0^2 \left\{x\left(x-3\right)^2-x\right\}dx$$

$$=2\int_0^2\{x^3-6x^2+8x\}\,dx$$

$$=2\int_0^2 x(x-2)(x-4)\,dx$$

$$=2\int_0^2 x(x-2)\{(x-2)-2\}\,dx$$

$$=2\int_0^2 x\left(x-2\right)^2 dx-4\int_0^2 x(x-2)\,dx$$

$$\frac{1}{6}\times 2^4+\frac{2}{3}\times 2^3=8$$